

Gaussian random waves in elastic media

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Similar to the Berry conjecture of quantum chaos we consider elastic analogue which incorporates longitudinal and transverse elastic displacements with corresponding wave vectors. Based on that we derive the correlation functions for amplitudes and intensities of elastic displacements. Comparison to numerics in a quarter Bunimovich stadium demonstrates excellent agreement.

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I. INTRODUCTION

Attracting interest in the field of wave chaos [1], elastomechanical systems are being studied analytically, numerically, and experimentally. Weaver first measured the few hundred lower eigen frequencies of an aluminum block and worked out the spectral statistics [2]. Spectral statistics coinciding with random matrix theory were observed in experiments for monocrystalline quartz blocks shaped as three-dimensional Sinai billiards [3], as well as, in experimental and numerical studies of flexural modes [4, 5] and in-plane modes [6, 7] for stadium-shaped plates. Statistical properties of eigen functions describing standing waves in elastic billiards were first reported by Schaadt *et al.* [8]. The authors measured the displacement field of several eigen modes of a thin plate shaped as a Sinai stadium. Due to a good preservation of up-down symmetry in the case of thin plates they dealt with two types of modes. The flexural modes with displacement perpendicular to the plane of the plate are well described by the scalar biharmonic Kirchhoff-Love equation [9, 10]. In this case a good agreement with theoretical prediction for both intensity statistical distribution and intensity correlation function was found. However in the case of in-plane displacements described by the vectorial Navier-Cauchy equation [9, 10] an agreement between the intensity correlator experimental data and the theory was not achieved [8].

The aim of present letter is to present an analogue of the Berry conjecture for elastic vibrating solids and derive the amplitude and intensity correlators with corresponding comparison to numerics. Quite recently, Acolzin and Weaver suggested a method to calculate the intensity correlator of vibrating elastic solids [11]. Based on the Green's function averaging technique they succeeded to derive the intensity correlator of flexural modes generalized due the finite thickness of a plate. Although the method might be used for the in-plane modes in elastic chaotic billiards, we propose here a more simple and physically transparent approach based on random superposition of traveling plane waves (Gaussian random wave (GRW) or the Berry function [12]). We show that the approach allows to derive all kinds of the correlators of RGW not only in infinite elastic media but also to take

into account the double ray splitting at the boundary of a plate that plays a significant role in the elastomechanical chaotic motion [13, 14]. We restrict ourselves to the two-dimensional case because of the current experiments available. Note however that the method can be easily generalized for the three-dimensional case.

II. ANALOGUE OF THE BERRY CONJECTURE IN ELASTIC MEDIA

Shapiro and Goelman [15] first presented statistics of the eigenfunctions in chaotic quantum billiard although their numerical histogram was not compared with the Gaussian distribution. This was done by McDonnell and Kaufmann [16] who concluded that the majority ($\approx 90\%$) of the eigenfunctions of the Bunimovich billiard are a Gaussian random field. Later it was confirmed by numerous numerical and experimental studies. The simple way to construct RGF is random superposition of particular solutions of Eq. (4) [12, 17, 18] with sufficient number N . Thus we come to the Berry conjecture in the form [1, 19]

$$\psi_B(\mathbf{x}) = \sqrt{\frac{1}{N}} \sum_{n=1}^N \exp[i(\theta_n + \mathbf{k}_n \mathbf{x})], \quad (1)$$

where the phases θ_n are random distributed uniformly in range $[0, 2\pi)$ and all the amplitudes are taken to be equal (one could assume random independent amplitudes, without any change in the results). The wave vectors \mathbf{k}_n are uniformly distributed on a d-dimensional sphere of radius k . It follows now from the central limit theorem that both $Re\psi_B$ and $Im\psi_B$ are independent Gaussian variables. In a closed billiard the Berry function is viewed as a sum of many standing waves, that is simply real or imaginary part of function (1).

In our case one has to construct a RGW-function describing acoustic in-plane modes. These modes are described by a two-dimensional Navier-Cauchy equation [10, 20]

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla(\nabla \mathbf{u}) + \rho \Omega^2 \mathbf{u} = 0 \quad (2)$$

where $\mathbf{u}(x, y)$ is the displacement field in the plate, λ, μ are the material dependent Lamé coefficients, and ρ is the density. Introducing elastic potentials ψ and \mathbf{A} with the help of the Helmholtz [20] decomposition the displacement field \mathbf{u} could be written,

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t, \quad \mathbf{u}_l = \nabla\psi, \quad \mathbf{u}_t = \nabla \times \mathbf{A} \quad (3)$$

Eq. (2) reduces to two Helmholtz equations for the elastic potentials

$$\begin{aligned} -\nabla^2\psi &= k_l^2\psi, \\ -\nabla^2\mathbf{A} &= k_t^2\mathbf{A}. \end{aligned} \quad (4)$$

Here $k_l = \omega/c_l$, $k_t = \omega/c_t$ are the wave numbers for the longitudinal and transverse waves, respectively and $\omega^2 = \rho\Omega^2/E$, where E is Young's modulus. In the two-dimensional case potential \mathbf{A} has only one non-zero component A_z and the dimensionless longitudinal and transverse sound velocities $c_{l,t}$ are given by

$$c_l^2 = \frac{1}{1-\sigma^2}, \quad c_t^2 = \frac{1}{2(1+\sigma)}, \quad (5)$$

where σ is Poisson's ratio [10, 20]. E and σ are functions of the Lamé coefficients [10, 20]. Our conjecture is that both elastic potential be statically independent Berry-like functions (1). We write the potentials in the following form

$$\begin{aligned} \psi(\mathbf{x}) &= \frac{a_l}{ik_l} \sqrt{\frac{1}{N}} \sum_{n=1}^N \exp[i(\mathbf{k}_{ln}\mathbf{x} + \theta_{ln})], \\ A_z(\mathbf{x}) &= \frac{a_t}{ik_t} \sqrt{\frac{1}{N}} \sum_{n=1}^N \exp[i(\mathbf{k}_{tn}\mathbf{x} + \theta_{tn})], \end{aligned} \quad (6)$$

where θ_{ln} , θ_{tn} are statistically independent random phases. The wave vectors \mathbf{k}_{ln} and \mathbf{k}_{tn} are uniformly distributed on circles of radii k_l and k_t respectively. According to (3) the components u, v of the vectorial displacement field \mathbf{u} could be now written

$$\begin{aligned} u(\mathbf{x}) &= \sqrt{\frac{1-\gamma}{N}} \sum_{n=1}^N \cos\phi_{ln} \exp[i(\mathbf{k}_{ln}\mathbf{x} + \theta_{ln})] \\ &\quad + \sqrt{\frac{\gamma}{N}} \sum_{n=1}^N \sin\phi_{tn} \exp[i(\mathbf{k}_{tn}\mathbf{x} + \theta_{tn})], \\ v(\mathbf{x}) &= \sqrt{\frac{1-\gamma}{N}} \sum_{n=1}^N \sin\phi_{ln} \exp[i(\mathbf{k}_{ln}\mathbf{x} + \theta_{ln})] \\ &\quad - \sqrt{\frac{\gamma}{N}} \sum_{n=1}^N \cos\phi_{tn} \exp[i(\mathbf{k}_{tn}\mathbf{x} + \theta_{tn})], \end{aligned} \quad (7)$$

where ϕ_{ln} , ϕ_{tn} are the angles between \mathbf{k}_{ln} , \mathbf{k}_{tn} and the x-axis respectively. The prefactors $a_l = \sqrt{\gamma}$, $a_t = \sqrt{1-\gamma}$ are chosen from the normalization condition $\langle \mathbf{u}^\dagger \mathbf{u} \rangle = 1$,

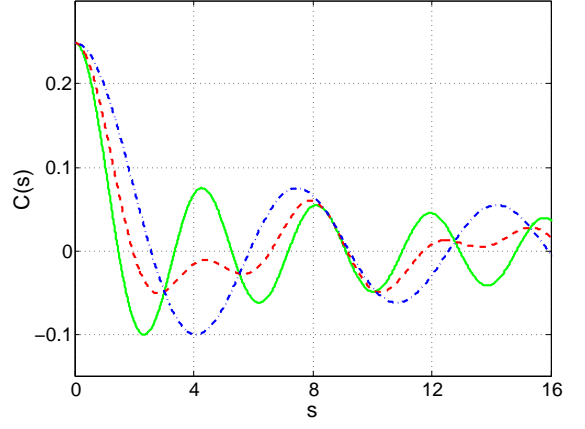


FIG. 1: Correlation function (10) for $\gamma = 0$ (green solid line), $\gamma = 0.5$ (red dashed line), and $\gamma = 1$ (blue dot-dashed line); $\sigma = 0.345$ (aluminum).

and $\langle \dots \rangle$ means average over the random phase ensembles. Parameter γ ranges from 0 (pure transverse waves) to 1 (pure longitudinal waves). By similar way one can construct the elastomechanical GRW for a closed system. One can see that the Berry analogue of chaotic displacements (7) is not a sum of two independent GRWs (or two independent Berry functions) $a_l\psi_l + a_t\psi_t$ as it was conjectured by Schaadt *et al.* [8] with arbitrary coefficients a_l , a_t . In fact, each component u and v in Eq (7) is related to the Berry functions (6) via space derivatives in accordance to relations (3).

III. CORRELATION FUNCTIONS

First, we calculate the amplitude correlation functions in chaotic elastic plate for in-plane GRW (7). For quantum mechanical GRW (1) the two-dimensional correlation function

$$\langle \psi_B(\mathbf{x} + \mathbf{s}) \psi_B^*(\mathbf{x}) \rangle = J_0(s)$$

was found firstly by Berry [12]. Straightforward procedure of averaging over ensembles of random phases θ_{ln} , θ_{tn} and next, over angles of k-vectors gives

$$\begin{aligned} \langle u(\mathbf{x} + \mathbf{s}) u(\mathbf{x}) \rangle &= \frac{\gamma}{2} (\cos^2\alpha f(k_ls) + \sin^2\alpha g(k_ls)) + \\ &\quad \frac{1-\gamma}{2} (\sin^2\alpha f(k_ts) + \cos^2\alpha g(k_ts)), \\ \langle v(\mathbf{x} + \mathbf{s}) v(\mathbf{x}) \rangle &= \frac{\gamma}{2} (\sin^2\alpha f(k_ls) + \cos^2\alpha g(k_ls)) + \\ &\quad \frac{1-\gamma}{2} (\cos^2\alpha f(k_ts) + \sin^2\alpha g(k_ts)), \\ \langle u(\mathbf{x} + \mathbf{s}) v(\mathbf{x}) \rangle &= \sin 2\alpha \left(\frac{1-\gamma}{2} J_2(k_ts) - \frac{\gamma}{2} J_2(k_ls) \right), \end{aligned} \quad (8)$$

where

$$f(s) = J_0(s) - J_2(s), \quad g(s) = J_0(s) + J_2(s). \quad (9)$$

It is important to note that the correlation functions (8) were obtained for given direction of the vector \mathbf{s} where

α is the angle included between vector \mathbf{s} and the x-axis. However, averaged over all directions of \mathbf{s} , the first two correlators simplify

$$\begin{aligned} C(s) &= \overline{\langle u(\mathbf{x} + \mathbf{s})u(\mathbf{x}) \rangle} = \overline{\langle v(\mathbf{x} + \mathbf{s})v(\mathbf{x}) \rangle} \\ &= \frac{1-\gamma}{2} J_0(k_l s) + \frac{\gamma}{2} J_0(k_t s) \end{aligned} \quad , \quad (10)$$

while the third vanishes $\overline{\langle u(\mathbf{x} + \mathbf{s})v(\mathbf{x}) \rangle} = 0$. One can see that in the averaged case the amplitude correlation function is defined by two scales because of two different sound velocities c_l , c_t ; that is obvious. The correlation function $C(s)$ is shown in Fig. 1.

Next, we calculate the intensity correlation functions $P(s) = \langle I(\mathbf{x} + \mathbf{s})I(\mathbf{x}) \rangle$ where the intensity $I = |\mathbf{u}|^2$ proportional to the elastic energy of the in-plane oscillations. In quantum mechanics this value is analogous to the probability density, the correlation function of which was calculated by Prigodin *et al.* [21]. For the in-plane chaotic GRW of the form $a_l\psi_l + a_t\psi_t$ Schaadt *et al.* [8] derived the intensity correlation function as

$$P(s) = 1 + 2[a_l^2 J_0(k_l s) + a_t^2 J_0(k_t s)]^2. \quad (11)$$

Our calculations similar to those as for the amplitude correlation functions (8) give the different result

$$\begin{aligned} P(s) &= 1 + \frac{1}{2\eta} [(\gamma J_0(k_l s) + (1-\gamma)J_0(k_t s))]^2 \\ &\quad + \frac{1}{2\eta} [(\gamma J_2(k_l s) - (1-\gamma)J_2(k_t s))]^2 \end{aligned} \quad (12)$$

where $\eta = 1$ for real GRW and $\eta = 2$ for complex one. Although the first term in (12) corresponds to (11) there is a different term consisted of the Bessel functions J_2 . The mathematical origin of deviation is that formula (7) contains the contributions of the components of the wave vectors \mathbf{k}_l and \mathbf{k}_t via space derivatives.

IV. WAVE CONVERSION AT BOUNDARY

Waves propagate freely inside the billiard, that is, the longitudinal and transverse components are decoupled. Wave conversion occurs at the boundary according to Snell's law

$$c_l \sin(\theta_t) = c_t \sin(\theta_l), \quad (13)$$

The reflection amplitudes for each event of the reflection can be easily found following the procedure described in [10]. At first we consider more easy case of the Dirichlet BC (the boundary is fixed). Approximating the boundary as the straight lines for the wavelengths much less than the radius of curvature we have for the reflection amplitudes

$$\begin{aligned} t_{ll} &= \frac{\cos(\theta_t) \cos(\theta_l) - \sin(\theta_t) \sin(\theta_l)}{\cos(\theta_t) \cos(\theta_l) + \sin(\theta_t) \sin(\theta_l)}, \\ t_{tt} &= \frac{2 \sin(\theta_l) \cos(\theta_l)}{\cos(\theta_t) \cos(\theta_l) + \sin(\theta_t) \sin(\theta_l)}, \end{aligned}$$

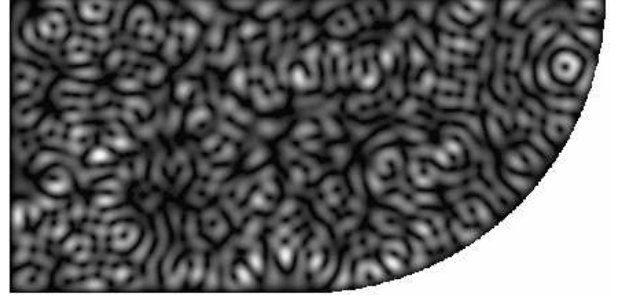


FIG. 2: The intensity $I = |\mathbf{u}|^2$ of the eigen state at frequency $\omega = 28.4$ in the quarter of the Bunimovich billiard with fixed boundary, $\sigma = 0.345$.

$$\begin{aligned} t_{ll} &= \frac{2 \sin(\theta_t) \cos(\theta_l)}{\cos(\theta_t) \cos(\theta_l) + \sin(\theta_t) \sin(\theta_l)}, \\ t_{tt} &= \frac{\cos(\theta_t) \cos(\theta_l) - \sin(\theta_t) \sin(\theta_l)}{\cos(\theta_t) \cos(\theta_l) + \sin(\theta_t) \sin(\theta_l)}. \end{aligned} \quad (14)$$

Next, we assume that all directions of waves are statistically equivalent. Then we have for the energy density of reflected wave

$$\rho_{out} = \gamma(\overline{T}_{ll} + \overline{T}_{lt}) + (1-\gamma)(\overline{T}_{tt} + \overline{T}_{tl}), \quad (15)$$

where

$$\overline{T}_{ij} = \frac{1}{\pi} \int_0^\pi t_{ij}^2 d\theta_i, \quad i = l, t.$$

Substituting into here (14) one can obtain after elementary calculations

$$\begin{aligned} \overline{T}_{ll} &= 1 - \frac{c_t}{c_l} I_1, \quad \overline{T}_{lt} = I_2, \\ \overline{T}_{tt} &= 1 - \frac{2}{\pi} \arcsin \frac{c_t}{c_l} + \left(\frac{c_t}{c_l} \right)^3 I_1, \\ \overline{T}_{tl} &= \frac{2}{\pi} \arcsin \frac{c_t}{c_l} - \left(\frac{c_t}{c_l} \right)^2 I_2. \end{aligned} \quad (16)$$

We do not present here integrals I_1, I_2 , since after substitution of (16) into (15) they cancel each other. The equality $\rho_{in} = 1 = \rho_{out}$ gives a very simple evaluation

$$\gamma = \frac{c_t^2}{c_t^2 + c_l^2}. \quad (17)$$

The next remarkable result is that although the reflection amplitudes for the free BC [10] have the form different from (14) (see, for example, formulas in [10]), the evaluation of γ by the same procedure gives the same form as for the fixed BC. Therefore, we can conclude that the result does not depend on either the free BC or the fixed BC is applied. Using (5) formula (17) could be written in a more simple form

$$\gamma = \frac{1 - \sigma}{3 - \sigma}. \quad (18)$$

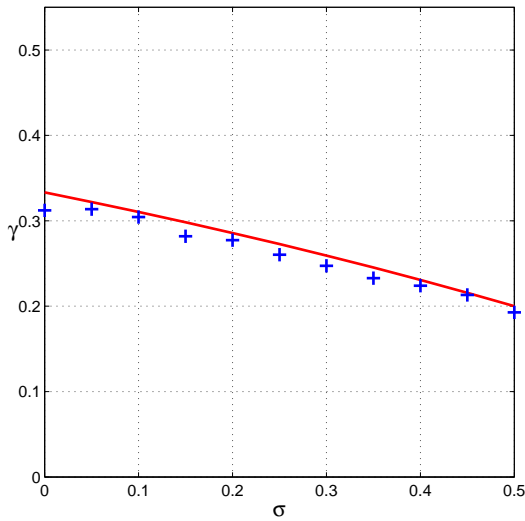


FIG. 3: Cross marks show numerical results for ratio γ averaged over 200 eigenfunctions of the quarter Bunimovich plate. The solid line is plotted by formula (18).

V. NUMERICAL RESULTS AND CONCLUSION

For numerical tests we took the quarter of the Bunimovich billiard and calculated the eigenstates of the Navier-Cauchy equation (2) with the fixed BC: $u = 0$, $v = 0$ at the boundary of the billiard by the finite-difference method. An example of the eigenstate in the form of intensity $I = |\mathbf{u}|^2$ is presented in Fig. 2. First of all we verified formula (18). For each value of Poisson's ratio σ in the range $[0, 0.5]$ with the step 0.05, 200 eigenfunctions of the billiard were found to calculate av-

eraged γ . The resulted dependence of γ on σ is shown in Fig. 3, that demonstrates a good agreement with formula (5). Thereby we can evaluate γ for specific $\sigma = 0.345$

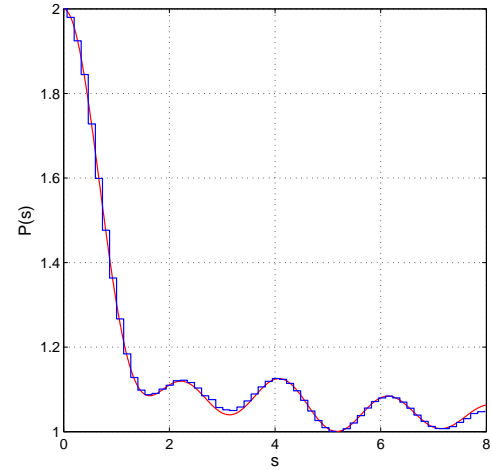


FIG. 4: The intensity correlation function (12) compared to the numerics for the same situation as in Fig. 2.

which correspond to aluminum plate and plot the correlation functions. The intensity correlation function (12) is shown in Fig. 4 compared to the numerics calculated for the eigen function presented in Fig. 2. One can see a good coincidence of the theory with the numerical results that demonstrates correctness of the GRW-approach (7) to chaos in elastic billiards.

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